

## Comparison of Gauss Quadrature and Newton–Cotes methods of order 10 in definite integral calculations

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**Received:** Jan 11, 2026 | **Revised:** Apr 10, 2026 | **Accepted:** Apr 16, 2026 | **Published Online:** Apr 28, 2026

### Abstract

*This study aims to evaluate and compare the accuracy of the Gauss Quadrature Method and the 10th Order Newton–Cotes Method based on numerical error in computing definite integrals, especially for complex functions that are difficult to solve analytically. The Gauss Quadrature Method studied includes the use of 2-point, 3-point, 4-point, 5-point, and 6-point rules, where the term points refers to the number of nodes used in the integration process. In contrast, the 10th-order Newton–Cotes Method is used for comparison, based on a high-degree interpolation polynomial at equidistant points. The comparison of the two methods is based on the accuracy of the numerical results as the main criterion. In this study, Microsoft Excel is used as a tool for numerical calculations. The results show that the Gauss Quadrature Method with 6 points provides numerical integral results with the highest level of accuracy compared to the Gauss Quadrature Method with fewer points, but generally produces larger errors than the 10th Order Newton–Cotes Method. These results indicate that, for the cases considered in this study, the 10th-order Newton–Cotes method is more efficient and accurate for calculating definite integrals.*

**Keywords:** Definite integral, Gauss quadrature, Newton–Cotes order 10, numerical methods.

**How to Cite:** Mulyono., Sumargo, Bagus., Suryana, M.E., Widyati, Ratna. (2026). Comparison of Gauss Quadrature and Newton–Cotes methods of order 10 in definite integral calculations. *Aksioma: Jurnal Matematika dan Pendidikan Matematika*, 17(1), 1-18. <https://doi.org/10.26877/jdxqqz70>

## INTRODUCTION

The definite integral is a fundamental concept in calculus that plays a vital role in various fields of science and technology. In applied mathematics, definite integrals are used to determine the area, volume, work, energy, and various other physical quantities. Furthermore, definite integrals are widely applied in engineering, economics, and computational science to model and analyze various real-world phenomena (Stewart, 2016). However, not all definite integrals can be solved analytically, especially when the functions involved are complex, discontinuous, or lack simple antiderivatives.

The limitations of analytical methods have led to the development of numerical methods as an alternative for calculating definite integrals. Numerical methods aim to obtain approximate solutions with a certain degree of accuracy through systematic computational procedures. Some commonly used numerical integration methods include the Trapezoidal, Simpson, Romberg, and Newton–Cotes methods. The Newton–Cotes

method uses an interpolation polynomial approach with equidistant evaluation points and is widely used due to its relatively simple concept and ease of implementation (Chapra & Canale, 2015; Davis & Rabinowitz, 2007). However, the use of high orders in the Newton–Cotes method often leads to oscillations and reduced numerical stability (Rao, 2002; Dahlquist & Björck, 1974).

Besides the Newton–Cotes method, one numerical integration method known for its high accuracy is the Gaussian Quadrature method. This method differs from conventional numerical integration methods in that it does not use a uniform division of the interval. Instead, the Gaussian Quadrature method determines optimal evaluation points and integration weights based on the roots of orthogonal polynomials, specifically the Legendre polynomials for the standard interval  $[-1,1]$  (Kiusalaas, 2005). With this approach, the Gaussian Quadrature method can exactly evaluate the integral of polynomials of degree up to  $2n-1$  using only  $n$  integration points.

The advantages of the Gaussian Quadrature method lie in its efficiency and accuracy. Compared to the Newton–Cotes method, the Gaussian Quadrature method generally requires fewer evaluation points to achieve the same or even better accuracy (Quarteroni et al., 2007). Therefore, this method is widely used in advanced numerical calculations, mathematical modeling, and computational simulations.

Several previous studies have shown that the Gaussian Quadrature method has faster convergence than classical numerical integration methods (Burden & Faires, 2011; Ralston & Rabinowitz, 1978; Powell, 1968), while the Newton–Cotes method is still widely used because of its ease of implementation (Davis & Rabinowitz, 2007). However, studies directly comparing the two methods, particularly at high Newton–Cotes orders, remain relatively limited, especially in simple implementations using readily accessible software.

Based on this, this study aims to compare the Gauss Quadrature method and the 10th-order Newton–Cotes method for calculating definite integrals. The Gauss Quadrature method is analyzed for variations in the number of integration points (2, 3, 4, 5, and 6), while the 10th Order Newton–Cotes method is used as a comparison method. The main focus of this study is to evaluate the level of accuracy of the numerical integration results produced by the two methods.

The novelty of this research lies in the systematic comparative analysis between the Gauss Quadrature and the 10th-order Newton–Cotes methods in terms of accuracy and error characteristics across different test cases. This study provides a clearer understanding of how each method performs under varying conditions, highlighting their respective strengths and limitations. In addition, the implementation using a widely accessible tool supports the reproducibility of the computational approach for educational and practical purposes.

## **METHODS**

This research is a numerical study that compares the accuracy of the Gaussian Quadrature method and the 10th-order Newton–Cotes method for calculating definite integrals. The research stages are as follows:

### **1. Test Function Selection**

Several test functions  $f(x)$  are selected for analysis. These functions include both analytically solvable and more complex forms. The selection of test functions is based on previous studies that have discussed the effectiveness of numerical integration methods (Kiusalaas, 2005; Quarteroni et al., 2007).

### **2. Integral Value Calculation**

For each test function, the integral value is calculated using several approaches, namely:

- a. Analytical method (if available) used as a reference value.
- b. Gauss Quadrature Method with variations in the number of integration points of 2, 3, 4, 5, and 6 points.
- c. Newton–Cotes method of order 10 with equidistant evaluation points.

It should be noted that the parameters used in the Newton–Cotes and Gauss Quadrature methods are not directly comparable. In the Newton–Cotes method, the parameter  $n$  denotes the number of partitions of the integration interval, whereas in Gauss Quadrature it denotes the number of evaluation points in a transformed interval. Therefore, the comparison in this study is based on the resulting accuracy rather than equivalent parameter settings.

The implementation of both numerical methods follows standard formulations widely used in the numerical analysis literature (Burden & Faires, 2011; Davis & Rabinowitz, 2007; Press et al., 2007).

### 3. Numerical Computation

The numerical calculations were performed using Microsoft Excel. The results obtained from each method were then compared with analytical values as a reference (Chapra & Canale, 2015).

### 4. Accuracy Evaluation

Accuracy evaluation is performed by calculating the error between the numerical result and the reference value. The error parameters used include:

- Absolute Error =  $|\text{Numerical Results} - \text{Analytical Results}| \times 100\%$
- Relative Error =  $\frac{|\text{Numerical results} - \text{Analytical Results}|}{\text{Analytical Results}} \times 100 \%$ .

The error parameters used in this study include absolute and relative errors, which are employed to evaluate and compare the accuracy of numerical methods against analytical results. Absolute error measures the direct difference, while relative error provides a normalized measure with respect to the analytical value. The use of both metrics allows for a more comprehensive assessment, as commonly adopted in numerical analysis literature (Süli & Mayers, 2003; Trefethen, 2000; Kelley, 1995).

### 5. Analysis of Results

The results of the numerical calculations were analyzed to determine the difference in accuracy between the Gauss Quadrature method and the 10th-Order Newton–Cotes method. In addition, the effect of the number of integration points on the Gauss Quadrature method was also analyzed to assess the balance between accuracy and calculation complexity (Yao, 2007; Triatmojo, 1992).

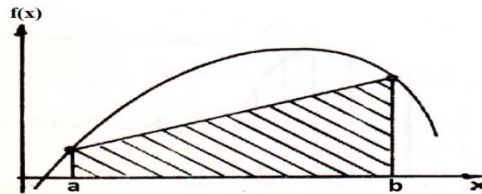
In this research, we discuss methods for approximating the definite integral  $\int_a^b f(x)dx$  when the function  $f(x)$  does not have an analytical (closed-form) solution or is difficult to integrate analytically, the methods considered are the Gauss Quadrature Method and the 10th Order Newton–Cotes Method (Stewart, 2016; Chapra & Canale, 2015).

#### 1. Quadrature Method

In a Trapezoidal or Simpson rule, the function being integrated numerically consists of two forms: a data table and a function. In the Quadrature method, specifically

the Gaussian Quadrature method, the function being integrated is a function (Atkinson, 1989; Steven, 2007; Sianipar, 2013; Zhang & Jin, 1996).

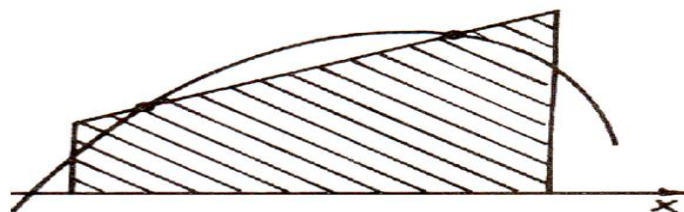
In the Trapezoidal and Simpson's rules, the integral is based on the values at the ends of the interval. As in the following figure, the Trapezoidal rule is based on the area under the straight line connecting the values of the function at the ends of the interval of integration (Triatmojo, 1992).



**Figure 1.** Graphical form of the Trapezoid rule

The formula used to calculate the area is:  $I = (b - a) \frac{f(a)+f(b)}{2}$ , a and b are the limits of the integration, and  $b - a =$  the width of the integration interval. Because the Trapezoid rule must pass through the endpoints, as shown in the image above, the Trapezoid formula yields a fairly large error (Triatmojo, 1992; Powell, 1968).

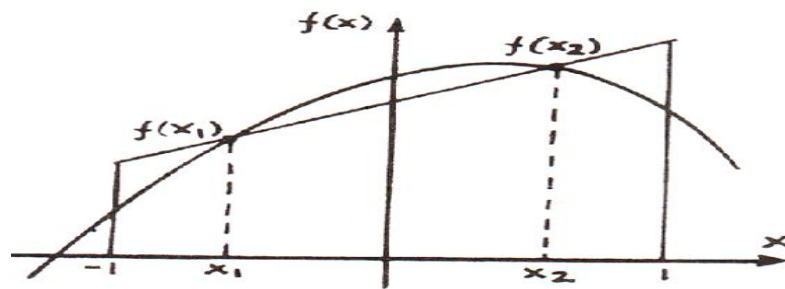
In the Gaussian Quadrature method, the area under a straight line connecting two arbitrary points on a curve is calculated. By freely positioning the two points, a straight line can be determined that balances positive and negative errors, as shown in the following figure (Triatmojo, 1992; Rao, 2002):



**Figure 2.** Graphical form of Gauss's Quadrature rule

In the Trapezoidal rule, the formula:  $I = (b - a) \frac{f(a)+f(b)}{2}$  can be written in the form :  $I = c_1f(a) + c_2f(b)$ , and the coefficients  $c_1$  and  $c_2$  will be sought. In the Gauss Quadrature method, the coefficients of the equation in the form  $I = c_1f(x_1) + c_2f(x_2)$  will also be sought.

In the equation above,  $x_1$  and  $x_2$  are not fixed and will be searched for, as in the following image:



**Figure 3.** Gaussian Quadrature Integration

So the Gaussian Quadrature equation:  $I = c_1f(x_1) + c_2f(x_2)$ , contains 4 unknown numbers, namely  $c_1$ ,  $c_2$ ,  $x_1$  and  $x_2$ , so 4 equations are needed to solve it. Thus, the equation:  $I = c_1f(x_1) + c_2f(x_2)$  is considered to satisfy the integral of the following 4 functions:  $f(x) = 1$ ,  $f(x) = x$ ,  $f(x) = x^2$ , and  $f(x) = x^3$  (Triatmojo, 1992).

So we get 4 equations:

$$f(x)=1, \text{ then } c_1f(x_1) + c_2f(x_2) = c_1 \cdot 1 + c_2 \cdot 1 = c_1 + c_2 = \int_{-1}^1 1 \, dx = 2$$

$$f(x)=x, \text{ then } c_1f(x_1) + c_2f(x_2) = c_1x_1 + c_2x_2 = \int_{-1}^1 x \, dx = 0$$

$$f(x)=x^2, \text{ then } c_1f(x_1) + c_2f(x_2) = c_1x_1^2 + c_2x_2^2 = \int_{-1}^1 x^2 \, dx = \frac{2}{3}$$

$$f(x)=x^3, \text{ then } c_1f(x_1) + c_2f(x_2) = c_1x_1^3 + c_2x_2^3 = \int_{-1}^1 x^3 \, dx = 0$$

Obtained a system of equations:

$$c_1 + c_2 = 2.$$

$$c_1x_1 + c_2x_2 = 0$$

$$c_1x_1^2 + c_2x_2^2 = \frac{2}{3}$$

$$c_1x_1^3 + c_2x_2^3 = 0$$

By solving the system of equations analytically using the elimination and substitution methods, the coefficients and integration points that satisfy the system are obtained as follows:  $c_1 = c_2 = 1$ ,  $x_1 = -\frac{1}{\sqrt{3}} = -0,577350269$ ,  $x_2 = \frac{1}{\sqrt{3}} = 0,577350269$ . From these results, the Gauss Quadrature formula (specifically Gauss-Legendre) is produced as follows:  $I = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$ , which is the Gauss Quadrature formula for 2 points (Triatmojo, 1992).

The integral limits used to derive the Gauss Quadrature formula are  $x = -1$  to  $x = 1$ , as this standard interval simplifies the computation and allows the formula to be applied more generally. To evaluate integrals with arbitrary limits  $[a,b]$ , a variable transformation is introduced. A linear relationship between the original variable  $x$  and a transformed

variable  $x_d$ , defined over the interval  $[-1,1]$ , is assumed in the form  $x = a_0 + a_1x_d$ . By applying the boundary conditions  $x = a$  when  $x_d = -1$  and  $x = b$  when  $x_d = 1$ , the coefficients are obtained as  $a_0 = \frac{b+a}{2}$  and  $a_1 = \frac{b-a}{2}$ . Thus, the transformation becomes  $x = \frac{(b+a)+(b-a)x_d}{2}$  and  $dx = \frac{b-a}{2} dx_d$ . These expressions are then substituted into the original integral to transform it from the interval  $[a,b]$  into the standard interval  $[-1,1]$ .

Gauss's Quadrature Formula for 2 points can be extended to n points as follows:  $I = c_1f(x_1) + c_2f(x_2) + \dots + c_nf(x_n)$ .

The values of c and x for the formula up to 6 points, as in the following table(Stewart, 2016; Triatmojo, 1992):

**Table 1.** Values of c and x in the Gaussian Quadrature formula

Number of points	Coefficient c	Variable x
2	$c_1 = 1,000000000$	$x_1 = -0,577350269$
	$c_2 = 1,000000000$	$x_2 = 0,577350269$
3	$c_1 = 0,555555556$	$x_1 = -0,774596669$
	$c_2 = 0,888888889$	$x_2 = 0,000000000$
	$c_3 = 0,555555556$	$x_3 = 0,774596669$
4	$c_1 = 0,347854845$	$x_1 = -0,861136312$
	$c_2 = 0,652145155$	$x_2 = -0,339981044$
	$c_3 = 0,652145155$	$x_3 = 0,339981044$
	$c_4 = 0,347854845$	$x_4 = 0,861136312$
5	$c_1 = 0,236926885$	$x_1 = -0,906179846$
	$c_2 = 0,478628670$	$x_2 = -0,538469310$
	$c_3 = 0,568888889$	$x_3 = 0,000000000$
	$c_4 = 0,478628670$	$x_4 = 0,538469310$
	$c_5 = 0,236926885$	$x_5 = 0,906179846$
6	$c_1 = 0,171324492$	$x_1 = -0,932469514$
	$c_2 = 0,360761573$	$x_2 = -0,661209386$
	$c_3 = 0,467913935$	$x_3 = -0,238619186$
	$c_4 = 0,467913935$	$x_4 = 0,238619186$
	$c_5 = 0,360761573$	$x_5 = 0,661209386$
	$c_6 = 0,171324492$	$x_6 = 0,932469514$

## 2. Newton Cotes Method

Simpson 1/3 Method and Simpson 3/8 Method are two numerical integration methods of the Newton-Cotes Method, namely Simpson 1/3 method is another name for the 2nd order Newton-Cotes method and Simpson 3/8 method is another name for the 3rd order Newton-Cotes method, where Simpson 1/3 method approximates the function  $f(x)$

with a 2nd degree interpolation polynomial and Simpson 3/8 method approximates the function  $f(x)$  with a 3rd degree interpolation polynomial. The general form of the Newton-Cotes method can be written as follows :  $\int_a^b f(x) dx \approx ah[w_0f_0 + w_1f_1 + w_2f_2 + \dots + w_nf_n]$ , with :  $f_j = f(x_j)$ ,  $x_j = a + jh$  and  $h = \frac{b-a}{n}$ , while  $\alpha$  and  $w_i$  are real constants and  $n$  is the order, as in the Table 2.

**Table 2.** Order (n) values and real constants of the Newton-Cotes method

$n$	$\alpha$	$w_i$									
1	$\frac{1}{2}$	1	1								
2	$\frac{1}{3}$	1	4	1							
3	$\frac{3}{8}$	1	3	3	1						
4	$\frac{2}{45}$	7	32	12	32	7					
5	$\frac{5}{288}$	19	75	50	50	75	19				
6	$\frac{1}{140}$	41	216	27	272	27	216	41			
7	$\frac{7}{17280}$	751	3577	1323	2989	2989	1323	3577	751		
8	$\frac{8}{14175}$	989	5888	-928	10496	-4540	10496	-928	58888	989	
9	$\frac{9}{89600}$	2857	15741	1080	19344	5788	5788	19344	1080		
10	$\frac{5}{299376}$	16067	106300	-48525	272400	-260550	427368				
		-260550	272400	-48525	106300	16067					

Based on Table 2, it can be observed that the coefficients  $\alpha$  and weights  $w_i$  vary depending on the order  $n$  of the Newton-Cotes method. For lower-order formulas, the weights are relatively simple and symmetric, whereas for higher-order formulas, they become more complex and may include negative values. This indicates that higher-order Newton-Cotes methods involve more intricate interpolation polynomials, which may affect numerical stability and accuracy.

## RESULTS AND DISCUSSION

In this research, we will be given several definite integral problems that can be solved analytically, as well as certain integral problems that are quite complicated, as follows:

1. Calculate  $\int_1^4 3 e^{0,1x^3} x^2 dx$  using analytical methods, Gauss Quadrature, and Newton-Cotes of order 10 using  $n = 30$  and  $60$ .
2. Calculate  $\int_3^6 e^{\sqrt{0,2x}} x^{5/3} (2x^3 - 3) dx$  using the Gauss Quadrature and Newton-Cotes method of order 10 using  $n = 30$ .

It should be noted that the parameters used in the Newton–Cotes and Gauss Quadrature methods are not directly comparable. In the Newton–Cotes method, the parameter  $n$  denotes the number of partitions of the integration interval, whereas in Gauss Quadrature it denotes the number of evaluation points in a transformed interval. Therefore, the comparison in this study is based on the resulting accuracy rather than equivalent parameter settings. Furthermore, the selected test functions span different levels of complexity, including one with an analytical solution and another without a simple analytical form. These cases are intended to provide comparative insight into the behavior of the numerical methods under different conditions. Therefore, the results of this study are limited to the tested cases and are not intended to be generalized.

The answers to the questions above are as follows:

### Answer to question number 1:

- a. The analytical answer to  $\int_1^4 3e^{0,1x^3} x^2 dx$  is:

$$\int 3e^{0,1x^3} x^2 dx = \int e^u 10 du = 10e^u + C = 10 e^{0,1x^3} + C$$

$$\int_1^4 3e^{0,1x^3} x^2 dx = 10 e^{0,1x^3} \Big|_1^4 = 10 e^{6,4} - 10 e^{0,1} = 6007,398669540$$

- b. The numerical answer of  $3 \int_1^4 e^{0,1x^3} x^2 dx$  using the Gaussian Quadrature method is :

It is known that the lower limit of the integral (a) = 1 and the upper limit of the integral (b) = 4. Variable substitution is carried out so that the integral limit changes to -1 to 1 as follows:

$$\mathbf{x} = \frac{(\mathbf{b}+\mathbf{a})+(\mathbf{b}-\mathbf{a})\mathbf{x}_d}{2} = \frac{(4+1)+(4-1)\mathbf{x}_d}{2} = 2,5 + 1,5\mathbf{x}_d \text{ so that } dx = 1,5 dx_d$$

$$\text{So : } \int_1^4 3e^{0,1x^3} x^2 dx = \int_{-1}^1 3 e^{0,1(2,5+1,5x_d)^3} (2,5 + 1,5x_d)^2 1,5 dx_d \text{ , so } \mathbf{f}(\mathbf{x}_d) = 3 e^{0,1(2,5+1,5x_d)^3} (2,5 + 1,5x_d)^2 1,5.$$

- Gauss Quadrature Method with 2 points:

For  $\mathbf{x}_d = \mathbf{x}_1 = -\frac{1}{\sqrt{3}}$ , then  $\mathbf{f}(\mathbf{x}_d) = \mathbf{f}(\mathbf{x}_1) = \mathbf{f}\left(\frac{-1}{\sqrt{3}}\right) = f(-0,577350269) = 18,585073501$

For  $\mathbf{x}_d = \mathbf{x}_2 = \frac{1}{\sqrt{3}}$ , then  $\mathbf{f}(\mathbf{x}_d) = \mathbf{f}(\mathbf{x}_2) = \mathbf{f}\left(\frac{1}{\sqrt{3}}\right) = f(0,577350269) = 2310,668646167$

So  $\int_{-1}^1 3 e^{0,1(2,5+1,5x_d)^3} (2,5 + 1,5x_d)^2 1,5 dx_d = c_1f(x_1) + c_2f(x_2) = f(x_1) + f(x_2) = 18,585073501 + 2310,668646167 = 2329,253719669$ , and the error is  $= \frac{|6007,398669540 - 2329,253719669|}{6007,398669540} = 61,2269162 \%$ .

- Gauss Quadrature Method with 3 points:

For  $\mathbf{x}_d = \mathbf{x}_1 = -0,774596669$ , then  $\mathbf{f}(\mathbf{x}_d) = \mathbf{f}(\mathbf{x}_1) = f(-0,774596669) = 0,238727066$

For  $\mathbf{x}_d = \mathbf{x}_2 = 0$ , then  $\mathbf{f}(\mathbf{x}_d) = \mathbf{f}(\mathbf{x}_2) = f(0) = 134,176870743$

For  $\mathbf{x}_d = \mathbf{x}_3 = 0,774596669$ , then  $\mathbf{f}(\mathbf{x}_d) = \mathbf{f}(\mathbf{x}_3) = f(0,774596669) = 188,187951002$

So  $\int_{-1}^1 3 e^{0,1(2,5+1,5x_d)^3} (2,5 + 1,5x_d)^2 1,5 dx_d = c_1f(x_1) + c_2f(x_2) + c_3f(x_3) = 0,555555556f(x_1) + 0,888888889f(x_2) + 0,555555556f(x_3) = 4673,949821023$ , and the error is  $= \frac{|6007,398669540 - 4673,949821023|}{6007,398669540} = 22,1967764 \%$ .

- Gauss Quadrature Method with 4 points:

For  $x_d = x_1 = -0,861136312$ , then  $f(x_d) = f(x_1) = f(-0,861136312) = 7,837401421$

For  $\mathbf{x}_d = \mathbf{x}_2 = -0,339981044$ , then  $\mathbf{f}(\mathbf{x}_d) = \mathbf{f}(\mathbf{x}_2) = f(-0,339981044) = 39,191854501$

For  $\mathbf{x}_d = \mathbf{x}_3 = 0,339981044$ , then  $\mathbf{f}(\mathbf{x}_d) = \mathbf{f}(\mathbf{x}_3) = f(0,339981044) = 623,252340337$

For  $\mathbf{x}_d = \mathbf{x}_4 = 0,861136312$ , then  $\mathbf{f}(\mathbf{x}_d) = \mathbf{f}(\mathbf{x}_4) = f(0,861136312) = 15078,887061331$

So  $\int_{-1}^1 e^{0,1(3,5+1,5x_d)^2} (3,5 + 1,5x_d) 1,5 dx_d = c_1f(x_1) + c_2f(x_2) + c_3f(x_3) + c_4f(x_4) = 0,347854845f(x_1) + 0,652145155f(x_2) + 0,652145155f(x_3) + 0,347854845f(x_4) = 5679,999971670$ , and the error is  $= \frac{|6007,398669540 - 5679,999971670|}{6007,398669540} = 5,4499246 \%$

- Gauss Quadrature Method with 5 points :

For  $x_d = x_1 = -0,906179846$ , then  $f(x_d) = f(x_1) = f(-0,906179846) = 6,792736662$

For  $x_d = x_2 = -0,538469310$ , then  $f(x_d) = f(x_2) = f(-0,538469310) = 20,924081015$

For  $x_d = x_3 = 0$ , then  $f(x_d) = f(x_3) = f(0) = 134,176870743$

For  $x_d = x_4 = 0,538469310$ , then  $f(x_d) = f(x_4) = f(0,538469310) = 1836,310397623$

For  $x_d = x_5 = 0,906179846$ , then  $f(x_d) = f(x_5) = f(0,906179846) = 21015,500057324$

So  $\int_{-1}^1 e^{0,1(3,5+1,5x_d)^2} (3,5 + 1,5x_d) 1,5 dx_d = c_1f(x_1) + c_2f(x_2) + c_3f(x_3) + c_4f(x_4) + c_5f(x_5) = 0,236926885f(x_1) + 0,47862867f(x_2) + 0,568888889f(x_3) + 0,47862867f(x_4) + 0,236926885f(x_5) = 5946,003746552$ , and the error is  $= \frac{|6007,398669540 - 5946,003746552|}{6007,398669540} = 1,0219885 \%$ .

- Gauss Quadrature Method with 6 points :

For  $x_d = x_1 = -0,932469514$ , then  $f(x_d) = f(x_1) = f(-0,932469514) = 6,237772483$

For  $x_d = x_2 = -0,661209386$ , then  $f(x_d) = f(x_2) = f(-0,661209386) = 14,424789305$

For  $x_d = x_3 = -0,238619186$ , then  $f(x_d) = f(x_3) = f(-0,238619186) = 55,174815234$

For  $x_d = x_4 = 0,238619186$ , then  $f(x_d) = f(x_4) = f(0,238619186) = 379,387635566$

For  $x_d = x_5 = 0,661209386$ , then  $f(x_d) = f(x_5) = f(0,661209386) = 3875,411893256$

For  $x_d = x_6 = 0,932469514$ , then  $f(x_d) = f(x_6) = f(0,932469514) = 25625,813670071$

So  $\int_{-1}^1 e^{0,1(3,5+1,5x_d)^2} (3,5 + 1,5x_d) 1,5 dx_d = c_1f(x_1) + c_2f(x_2) + c_3f(x_3) + c_4f(x_4) + c_5f(x_5) + c_6f(x_6) = 0,171324492f(x_1) + 0,360761573f(x_2) + 0,467913935f(x_3) + 0,467913935f(x_4) + 0,360761573f(x_5) + 0,171324492f(x_6) = 5998,039618984$ , and the error is  $= \frac{|6007,398669540 - 5998,039618984|}{6007,398669540} = 0,1557921 \%$ .

Based on the results for the tested case, the 6-point Gauss Quadrature method yields a relative error of 0.1557921%, which is lower than those obtained using the 2-point to 5-point methods. This suggests that increasing the number of points tends to improve the accuracy for this particular integral.

- c. The numerical answer of  $\int_1^4 3e^{0,1x^3} x^2 dx$  using the 10th order Newton-Cotes method using  $n = 30$  or  $h=0.1$  is:

From the calculation results, the following data was obtained:

**Table 3.** Values of  $x_i$  and  $f(x_i)$  of the function  $f(x) = 3x^2e^{0,1x^3}$  with  $h = 0,1$

i	$x_i$	$f_i = f(x_i)$	i	$x_i$	$f_i = f(x_i)$	i	$x_i$	$f_i = f(x_i)$
0	1,0	3,315512754	11	2,1	33,401038026	22	3,2	813,799408133
1	1,1	4,146782151	12	2,2	42,111758854	23	3,3	1188,155312272
2	1,2	5,134874496	13	2,3	53,577571296	24	3,4	1766,160308417
3	1,3	6,315714026	14	2,4	68,851345218	25	3,5	2674,822295832
4	1,4	7,736557041	15	2,5	89,451247162	26	3,6	4130,195327340
5	1,5	9,459717357	16	2,6	117,593666618	27	3,7	6506,636454133
6	1,6	11,56773257	17	2,7	156,556322699	28	3,8	1046,238471317
7	1,7	14,17059143	18	2,8	211,251871006	29	3,9	17196,572934233
8	1,8	17,41593547	19	2,9	289,146581703	30	4,0	28888,561817860
9	1,9	21,50359351	20	3,0	401,752756572	-	-	-
10	2,0	26,70649114	21	3,1	567,089143899	-	-	-

10th-Order Newton-Cotes Method Using  $n = 30$  :

Using the 10th-Order Newton-Cotes method and the data from Table 3, then :

$$\int_1^4 3 e^{0,1x^3} x^2 dx = \frac{5}{299376} * 0,1[16067f_0 + 106300f_1 - 48525f_2 + 272400f_3 - 260550f_4 + 427368f_5 - 260550f_6 + 272400f_7 - 48525f_8 + 106300f_9 + 32134f_{10} + 106300f_{11} - 48525f_{12} + 272400f_{13} - 260550f_{14} + 427368f_{15} - 260550f_{16} + 272400f_{17} - 48525f_{18} + 106300f_{19} + 32134f_{20} + 106300f_{21} - 48525f_{22} + 272400f_{23} - 260550f_{24} + 427368f_{25} - 260550f_{26} + 272400f_{27} - 48525f_{28} + 106300f_{29} + 16067f_{30}] = 6007,402365551.$$

and the error =  $\frac{|6007,398669540 - 6007,402365551|}{6007,398669540} = 0,000000615 = 0,0000615\%$ .

**Table 4.** Values of  $x_i$  and  $f(x_i)$  of the function  $f(x) = 3x^2e^{0,1x^3}$  with  $h = 0,05$

i	$x_i$	$f_i = f(x_i)$	i	$x_i$	$f_i = f(x_i)$	i	$x_i$	$f_i = f(x_i)$
0	1,00	3.315512754	21	2.05	29.838645284	42	3.1	567.089143899
1	1.05	3.713426803	22	2.10	33.401038026	43	3.15	677.903328434
2	1.10	4.146782151	23	2.15	37.464187360	44	3.2	813.799408133
3	1.15	4.619209901	24	2.20	42.111758854	45	3.25	981.159250597
4	1.20	5.134874496	25	2.25	47.443295601	46	3.3	1188.155312272
5	1.25	5.698550768	26	2.30	53.577571296	47	3.35	1445.297526931
6	1.30	6.315714026	27	2.35	60.656720974	48	3.4	1766.160308417
7	1.35	6.992645638	28	2.40	68.851345218	49	3.45	2168.353170370
8	1.40	7.736557041	29	2.45	78.366836793	50	3.5	2674.822295832
9	1.45	8.555735712	30	2.50	89.451247162	51	3.55	3315.603693333
10	1.50	9.459717357	31	2.55	102.405099074	52	3.6	4130.195327340
11	1.55	10.459489480	32	2.60	117.593666618	53	3.65	5170.781537418
12	1.60	11.567732571	33	2.65	135.462394260	54	3.7	6506.636454133
13	1.65	12.799106509	34	2.70	156.556322699	55	3.75	8230.166041609
14	1.70	14.170591427	35	2.75	181.544646926	56	3.8	10465.238471316
15	1.75	15.701894354	36	2.80	211.251871006	57	3.85	13378.725636902
16	1.80	17.415935466	37	2.85	246.697472249	58	3.9	17196.572934233
17	1.85	19.339430964	38	2.90	289.146581703	59	3.95	22226.286556077
18	1.90	21.503593507	39	2.95	340.174979009	60	4	28888.561817860
19	1.95	23.944976056	40	3,00	401.752756572	-	-	-
20	2,00	26.706491142	41	3.05	476.352425463	-	-	-

10th-order Newton-Cotes method using  $n = 60$ :

Using the 10th-order Newton-Cotes method and the data from Table 4, then:

$$\int_1^4 3 e^{0,1x^3} x^2 dx = \frac{5}{299376} * 0,1[16067f_0 + 106300f_1 - 48525f_2 + 272400f_3 - 260550f_4 + 427368f_5 - 260550f_6 + 272400f_7 - 48525f_8 + 106300f_9 + 32134f_{10} + 106300f_{11} - 48525f_{12} + 272400f_{13} - 260550f_{14} + 427368f_{15} - 260550f_{16} + 272400f_{17} - 48525f_{18} + 106300f_{19} + 32134f_{20} + 106300f_{21} - 48525f_{22} + 272400f_{23} - 260550f_{24} + 427368f_{25} - 260550f_{26} + 272400f_{27} - 48525f_{28} + 106300f_{29} + 32134f_{30} + 106300f_{31} - 48525f_{32} + 272400f_{33} - 260550f_{34} + 427368f_{35} - 260550f_{36} + 272400f_{37} - 48525f_{38} + 106300f_{39} + 32134f_{40} + 106300f_{41} - 48525f_{42} + 272400f_{43} - 260550f_{44} + 427368f_{45} - 260550f_{46} + 272400f_{47} - 48525f_{48} + 106300f_{49} + 32134f_{50} + 106300f_{51} - 48525f_{52} + 272400f_{53} - 260550f_{54} + 427368f_{55} - 260550f_{56} + 272400f_{57} - 48525f_{58} + 106300f_{59} + 16067f_{60}] = 6007,398672479$$

$$\text{and the error} = \frac{|6007,398669540 - 6007,398672479|}{6007,398669540} = 0,000000000489240$$

$$= 0,0000000489240\%$$

Based on the results for the tested case, the 10th-order Newton–Cotes method with a partition width of  $h=0.05$  (60 partitions) yields a smaller relative error of 0,0000000489240% than a partition width of  $h=0.1$  (30 partitions). This suggests that reducing the partition size may improve accuracy under the given conditions.

Answer to question number 2:

- a. The numerical integral of  $\int_3^6 e^{\sqrt{0,2x}} x^{5/3} (2x^3 - 3) dx$  using the Gaussian Quadrature method is as follows:

It is known that the lower limit of the integral (a) = 3 and the upper limit of the integral (b) = 6. Variable substitution is carried out so that the integral limit changes to -1 to 1 as follows :

$$x = \frac{(b+a)+(b-a)x_d}{2} = \frac{(6+3)+(6-3)x_d}{2} = 4,5 + 1,5x_d \text{ so that } dx = 1,5 dx_d$$

$$\text{So : } \int_3^6 e^{\sqrt{0,2x}} x^{5/3} (2x^3 - 3) dx = \int_{-1}^1 e^{\sqrt{0,2(4,5+1,5x_d)}} (4,5 + 1,5x_d)^{5/3} (2(4,5 + 1,5x_d)^3 - 3) 1,5 dx_d, \text{ so that } f(x_d) = e^{\sqrt{0,2(4,5+1,5x_d)}} (4,5 + 1,5x_d)^{5/3} (2(4,5 + 1,5x_d)^3 - 3) 1,5.$$

- Gauss Quadrature Method with 2 points :

For  $x_d = x_1 = -\frac{1}{\sqrt{3}}$ , then  $f(x_d) = f(x_1) = f\left(\frac{-1}{\sqrt{3}}\right) = f(-0,577350269) = 1722,926976710$

For  $x_d = x_2 = \frac{1}{\sqrt{3}}$ , then  $f(x_d) = f(x_2) = f\left(\frac{1}{\sqrt{3}}\right) = f(0,577350269) = 12911,421587182$

So  $\int_{-1}^1 e^{\sqrt{0,2(4,5+1,5x_d)}} (4,5 + 1,5x_d)^{5/3} (2(4,5 + 1,5x_d)^3 - 3) 1,5 dx_d = c_1 f(x_1) + c_2 f(x_2) = f(x_1) + f(x_2) = 1722.926976710 + 12911.421587182 = 14634,348563893.$

- Gauss Quadrature Method with 3 points :

For  $x_d = x_1 = -0,774596669$ , then  $f(x_d) = f(x_1) = f(-0,774596669) = 1114,850801115$

For  $x_d = x_2 = 0$ , then  $f(x_d) = f(x_2) = f(0) = 5172,142278453$

For  $x_d = x_3 = 0,774596669$ , then  $f(x_d) = f(x_3) = f(0,774596669) = 17109,289146262$

So  $\int_{-1}^1 e^{\sqrt{0,2(4,5+1,5x_d)}} (4,5 + 1,5x_d)^{5/3} (2(4,5 + 1,5x_d)^3 - 3) 1,5 dx_d = c_1f(x_1) + c_2f(x_2) + c_3f(x_3) = 0,555555556f(x_1) + 0,888888889f(x_2) + 0,555555556f(x_3) = 14721,982004731.$

- Gauss Quadrature Method with 4 points :

For  $x_d = x_1 = -0,861136312$ , then  $f(x_d) = f(x_1) = f(-0,861136312) = 909,663564305$

For  $x_d = x_2 = -0,339981044$ , then  $f(x_d) = f(x_2) = f(-0,339981044) = 2783,442832071$

For  $x_d = x_3 = 0,339981044$ , then  $f(x_d) = f(x_3) = f(0,339981044) = 9023,765839484$

For  $x_d = x_4 = 0,861136312$ , then  $f(x_d) = f(x_4) = f(0,861136312) = 19277,413516603$

So  $\int_{-1}^1 e^{\sqrt{0,2(4,5+1,5x_d)}} (4,5 + 1,5x_d)^{5/3} (2(4,5 + 1,5x_d)^3 - 3) 1,5 dx_d = c_1f(x_1) + c_2f(x_2) + c_3f(x_3) + c_4f(x_4) = 0,347854845f(x_1) + 0,652145155f(x_2) + 0,652145155f(x_3) + 0,347854845f(x_4) = 14722,186498211.$

- Gauss Quadrature Method with 5 points :

For  $x_d = x_1 = -0,906179846$ , then  $f(x_d) = f(x_1) = f(-0,906179846) = 815.538242639$

For  $x_d = x_2 = -0,538469310$ , then  $f(x_d) = f(x_2) = f(-0,538469310) = 1869.481116703$

For  $x_d = x_3 = 0$ , then  $f(x_d) = f(x_3) = f(0) = 5172.142278453$

For  $x_d = x_4 = 0,538469310$ , then  $f(x_d) = f(x_4) = f(0,538469310) = 12194.138643654$

For  $x_d = x_5 = 0,906179846$ , then  $f(x_d) = f(x_5) = f(0,906179846) = 20492.979345004$

So  $\int_{-1}^1 e^{\sqrt{0,2(4,5+1,5x_d)}} (4,5 + 1,5x_d)^{5/3} (2(4,5 + 1,5x_d)^3 - 3) 1,5 dx_d = c_1f(x_1) + c_2f(x_2) + c_3f(x_3) + c_4f(x_4) + c_5f(x_5) = 0,236926885f(x_1) + 0,47862867f(x_2) + 0,568888889f(x_3) + 0,47862867f(x_4) + 0,236926885f(x_5) = 14722,186591833.$

- Gauss Quadrature Method with 6 points :

For  $x_d = x_1 = -0,932469514$ , then  $f(x_d) = f(x_1) = f(-0,932469514) = 764,309153654$

For  $x_d = x_2 = -0,661209386$ , then  $f(x_d) = f(x_2) = f(-0,661209386) = 1438,276743820$

For  $x_d = x_3 = -0,238619186$ , then  $f(x_d) = f(x_3) = f(-0,238619186) = 3373,593165843$

For  $x_d = x_4 = 0,238619186$ , then  $f(x_d) = f(x_4) = f(0,238619186) = 7688,069770823$

For  $x_d = x_5 = 0,661209386$ , then  $f(x_d) = f(x_5) = f(0,661209386) = 14577,529984530$

For  $x_d = x_6 = 0,932469514$ , then  $f(x_d) = f(x_6) = f(0,932469514) = 21231,335991865$

So  $\int_{-1}^1 e^{\sqrt{0,2(4,5+1,5x_d)}} (4,5 + 1,5x_d)^{5/3} (2(4,5 + 1,5x_d)^3 - 3) 1,5 dx_d = c_1f(x_1) + c_2f(x_2) + c_3f(x_3) + c_4f(x_4) + c_5f(x_5) + c_6f(x_6) = 0,171324492f(x_1) + 0,360761573f(x_2) + 0,467913935f(x_3) + 0,467913935f(x_4) + 0,360761573f(x_5) + 0,171324492f(x_6) = 14722,186591291$ .

- b. The numerical answer of  $\int_3^6 e^{\sqrt{0,2x}} x^{5/3} (2x^3 - 3) dx$  using the 10th order Newton-Cotes method using  $n = 30$  or  $h = 0.1$  is :

From the calculation results, the following data is obtained :

**Table 5.** Values of  $x_i$  and  $f(x_i)$  of the function  $f(x) = e^{\sqrt{0,2x}} x^{5/3} (2x^3 - 3)$  with  $h = 0,1$

i	$x_i$	$f_i = f(x_i)$	i	$x_i$	$f_i = f(x_i)$	i	$x_i$	$f_i = f(x_i)$
0	1,0	3,315512754	11	2,1	33,401038026	22	3,2	813,799408133
1	1,1	4,146782151	12	2,2	42,111758854	23	3,3	1188,155312272
2	1,2	5,134874496	13	2,3	53,577571296	24	3,4	1766,160308417
3	1,3	6,315714026	14	2,4	68,851345218	25	3,5	2674,822295832
4	1,4	7,736557041	15	2,5	89,451247162	26	3,6	4130,195327340
5	1,5	9,459717357	16	2,6	117,593666618	27	3,7	6506,636454133
6	1,6	11,56773257	17	2,7	156,556322699	28	3,8	1046,238471317
7	1,7	14,17059143	18	2,8	211,251871006	29	3,9	17196,572934233
8	1,8	17,41593547	19	2,9	289,146581703	30	4,0	28888,561817860
9	1,9	21,50359351	20	3,0	401,752756572	-	-	-
10	2,0	26,70649114	21	3,1	567,089143899	-	-	-

10th Order Newton-Cotes Method using  $n = 30$ :

Using the 10th Order Newton-Cotes method and the data from Table 5:

$$\int_3^6 e^{\sqrt{0,2x}} x^{5/3} (2x^3 - 3) dx = \frac{5}{299376} * 0,1 [16067f_0 + 106300f_1 - 48525f_2 + 272400f_3 - 260550f_4 + 427368f_5 - 260550f_6 + 272400f_7 - 48525f_8 + 106300f_9 + 32134f_{10} + 106300f_{11} - 48525f_{12} + 272400f_{13} - 260550f_{14} + 427368f_{15} - 260550f_{16} + 272400f_{17} - 48525f_{18} + 106300f_{19} + 32134f_{20} + 106300f_{21} - 48525f_{22} + 272400f_{23} - 260550f_{24} + 427368f_{25} - 260550f_{26} + 272400f_{27} - 48525f_{28} + 106300f_{29} + 16067f_{30}] = 14722,186599817$$

The integral  $\int_3^6 e^{\sqrt{0,2x}} x^{5/3} (2x^3 - 3) dx$  does not have a simple analytical solution; therefore, numerical methods are applied. Based on the results for the tested case, the 10th-order Newton–Cotes method with a partition width of  $h=0.1$  and 30 partitions yields a value of 14722,186599817 and produces the smallest relative error among the methods used in this study.

## **CONCLUSION**

Based on the numerical analysis conducted in this study, both the Gauss Quadrature and Newton–Cotes methods are effective for approximating definite integrals. For the tested cases, increasing the number of points in the Gauss Quadrature method tends to improve the accuracy of the results. Similarly, in the Newton–Cotes method, smaller partition widths yield higher accuracy. Furthermore, among the methods considered, the 10th-order Newton–Cotes method with an appropriate partition size provides results with very small relative error for the tested cases. However, these findings are limited to the specific functions and conditions examined in this study.

## **DECLARATION**

Author Contributions: M: Conceptualization, Writing – Original Draft, Editing, and Visualization; BS, MES, and RW: Validation and Review.

Funding Declaration: This research was funded by the PNBP-BLU FUND from the Faculty of Mathematics and Natural Sciences, UNJ, Contract No.: 150 /PPK-FMIPA/BAP-PEN/2025, dated March 19, 2025.

Conflict of Interest: The authors declare no conflict of interest

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